Exploration 5
Introduction to Limits

1. Plot on your calculator the graph of this function.
   \[ f(x) = \frac{x^3 - 7x^2 + 17x - 15}{x - 3} \]
   Use a friendly window with \( x = 3 \) as a grid point. Sketch the results here. Show the behavior of the function in a neighborhood of \( x = 3 \).

2. Substitute 3 for \( x \) in the equation for \( f(x) \). What form does the answer take? What name is given to an expression of this form?

3. The graph of \( f \) has a removable discontinuity at \( x = 3 \). The \( y \)-value at this discontinuity is the limit of \( f(x) \) as \( x \) approaches 3. What number does this limit equal?

4. Make a table of values of \( f(x) \) for each 0.1 unit change in \( x \)-values from 2.5 through 3.5.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>3.1</td>
</tr>
<tr>
<td>2.6</td>
<td>3.2</td>
</tr>
<tr>
<td>2.7</td>
<td>3.3</td>
</tr>
<tr>
<td>2.8</td>
<td>3.4</td>
</tr>
<tr>
<td>2.9</td>
<td>3.5</td>
</tr>
</tbody>
</table>

5. Between what two numbers does \( f(x) \) stay when \( x \) is kept in the open interval (2.5, 3.5)?

6. Simplify the fraction for \( f(x) \). Solve numerically to find the two numbers close to 3 between which \( x \) must be kept if \( f(x) \) is to stay between 1.99 and 2.01.

7. How far from \( x = 3 \) (to the left and to the right) are the two \( x \)-values in Problem 6?

8. For the statement “If \( x \) is within _____ units of 3 (but not equal to 3), then \( f(x) \) is within 0.01 unit of 2,” write the largest number that can go in the blank.

9. Write the definition of limit.

10. Problem 8 gives four numbers that correspond to \( L, c, \) epsilon, and delta in the definition of limit. Which is which?

11. What did you learn as a result of doing this Exploration that you did not know before?
1. The function \( f(x) = \frac{4x^2 - 7x - 2}{x - 2} \) is undefined when \( x = 2 \). In this problem you will show that \( f(x) \) does have a limit as \( x \) approaches 2.
   
a. Plot the graph of \( f \). Use a friendly window that includes \( x = 2 \) as a grid point. Sketch the graph and name the feature that seems to be present at \( x = 2 \).
   
b. From the graph, tell what you think the limit of \( f(x) \) is as \( x \) approaches 2.
   
c. Try to evaluate \( f(2) \) by direct substitution. What form does the answer take? What name is given to a form such as this?
   
d. Factor the numerator and simplify the expression by canceling the common factor. Although the simplified expression does not equal \( f(2) \), you can substitute 2 for \( x \) and get an answer. What is this answer and what does it represent?
   
e. How close to 2 would you have to keep \( x \) in order for \( f(x) \) to be between 8.9 and 9.1?
   
f. How close to 2 would you have to keep \( x \) in order for \( f(x) \) to be within 0.001 unit of the limit in 1b? Answer in the form “\( x \) must be within _____ units of 2.”
   
g. Four constants appear in the definition of limit: \( L, c, \) epsilon, and delta. What are the values of these four constants in 1f?
   
h. Explain how you could find a suitable value of delta no matter how small epsilon is.
   
i. What is the reason for the restriction “…but not equal to \( c \)” that appears in the definition of limit?

2. The function \( f(x) = \frac{(x^2 - 6x + 13)(x - 2)}{(x - 2)} \) is undefined when \( x = 2 \). However, if you cancel the \( (x - 2) \) factors, the equation becomes \( f(x) = x^2 - 6x + 13 \) \( (x \neq 2) \). So \( f \) would be a quadratic function, except that there is a removable discontinuity where \( x = 2 \). The y-value of this missing point is the limit of \( f(x) \) as \( x \) approaches 2.
   
a. Show that \( f(2) \) has the indeterminate form 0/0. Do an appropriate calculation to show that 5 is the limit of \( f(x) \) as \( x \) approaches 2.
   
b. Plot the graph close to the discontinuity. Use a friendly window that includes \( x = 2 \) as a grid point and has an \( x \)-increment of 0.001. Then use your calculator to trace or the table feature to make a table of values of \( f(x) \) for each value of \( x \) from 1.990 through 2.010. For which values of \( x \) in the table is \( f(x) \) within 0.01 unit of 5? Complete the statement “If \( x \) is within _____ units of 2, then \( f(x) \) is within 0.01 unit of 5.”
   
c. Find the largest interval of values of \( x \) for which \( f(x) \) is within 0.01 unit of 5. You can do
this by setting \( f(x) = 4.99 \) and solving to find the value of \( x \) nearest 2. Repeat for \( f(x) = 5.01 \). Keep as much precision as your calculator will give you.

d. Sketch the part of the graph close to \( x = 2 \). Show how the numbers in 2c relate to the graph.

e. Find the largest number you could put in the blank of the statement in 2b. Take into account that the interval in 2c is of a different width on one side of 2 than on the other.

f. Write the values of the constants \( L, c, \epsilon, \) and \( \delta \) for 2a-e.

3. One-Sided Limit Problem: The following function has a step discontinuity at \( x = 2 \).

\[
f(x) = 3 + \frac{1}{2}x + \frac{|x - 2|}{x - 2}
\]

a. The figure below shows the graph of \( f \). Explain why the graph takes a jump at \( x = 2 \).

b. What is the limit of \( f(x) \) as \( x \) approaches 2 from the left side?

c. What is the limit of \( f(x) \) as \( x \) approaches 2 from the right side?

d. Explain why there is no single number that can be the limit of \( f(x) \) as \( x \) approaches 2.